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ORIENTATION STATISTICS IN THE EARTH SCIENCES

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1. INTRODUCTION

The Earth Sciences[1] seek to describe and understand the processes which have acted in the past, and are acting now, to form the continents and oceans and their mountains and valleys and which have lead to the varied sequences of rocks of differing compositions and structures. These processes in time move material from one point to another and so inevitably involve directional properties. It is thus natural that, to unravel these processes, measurements of directions should be made. This paper attempts to give statisticians a general view of the problems which arise, in the hope that others may become interested in the area. The paper thus contains more science than statistics. The coverage is more limited than the title, e.g. oceanography and meteorology have been excluded.

This writer's first point of contact with geologic problems was the relatively new subject of palaeomagnetism—see Irving (1964). This subject has made great contributions to theories of the origin of the earth's magnetic field and to the rehabilitation of the Continental Drift hypothesis. The basic data is the direction (not strength) of permanent magnetization of a rock specimen; this magnetism, it is supposed, was put in when the rock was laid down or last cooled down. It is discussed below in Section 4; statistical methods have been more actively developed for this area than any other. The rest of the paper is concerned with geological problems. A number of geological terms and concepts are defined below in this section so examples may be cited in Section 2 and 3. Section 2 discusses the field measurement of directional features with definitions of the many terms used. Section 3 contains a brief account of the relevant statistical methods. Sections 4, 5, 6, and 7 give some details of their applications to, respectively palaeomagnetism, sedimentary geology, structural geology and petrofabrics; difficulties and unsolved problems are mentioned.

The emphasis on directional problems varies greatly from one research area to another within geology although the general ideas were recognized in the early 19th century. Lava flows have internal (e.g. crystal orientation) and surface features which enable the top to be distinguished from the bottom and the direction of flow determined; but there is not, at the moment, a definitive book describing this area. Directional analysis has always been a major tool for the study of continental ice sheets whose direction of movement can be seen graphically in *striations* (scratches) on rocks, by the tendency of the long axes of boulders in till to be parallel to the movement, by terminal moraines to be normal to the movement, and by many other features. In this case however mapping of directions without refined statistical work, has sufficed to explain the process up to now. All the pieces of information are put together in terms of a "model"—a huge glacier advancing and retreating across a continent.

Sedimentary geologists make more sophisticated use of orientation studies which are nowhere better explained than in Potter & Pettijohn's (1963) book on palaeocurrents; much of what follows is drawn from this book. Sedimentary rocks were first studied with an eye to elucidating the time sequence of rock strata—initially by their fossil contents and later by other internal features. (Many readers will remember the references to Lyell's work in Sir Ronald Fisher's Presidential Address (1953) to the Royal Statistical Society.) Since these rocks are laid down during the transport of material by ocean currents, rivers and winds (aeolian deposits are rarely preserved), their structure should reveal the directions of ancient currents and slopes. If attention is directed to one large basin into which material is transported and laid down to form the sediments now seen as rocks, it should be possible

[1] Henceforth we shall use the word "geology" in its original sense which was what is now conveyed by "earth sciences".

to make a coherent account of the whole process—a Basin model. This model shows that various observed quantities will contribute to our understanding.

We note here, parenthetically, that geology is not just a study of the ancient history of the earth. Its study should lead to an understanding of the present and future of the earth. Furthermore it is necessary knowledge for the interpretation of data from the moon and other planets which is becoming available in increasing amounts. On the moon, radiation and the impacts of planetesimals have a much greater effect on surface features than on the earth.

Scalar quantities could have directional significance when mapped, e.g., larger pebbles will be deposited up stream of smaller pebbles so their size is such a scalar. The relative proportions of minerals can similarly be used although this has some vector characteristics mathematically. Thus statistical contributions to mapping should be included under our title; but we will be forced by time and space to concentrate mainly on intrinsically oriented quantities.

Many *vectors and tensors* are of geologic importance. The vectors are usually *unit vectors*, e.g. the direction of the palaeocurrent because its velocity is not known, the long axis of a fossil, from its blunt end to its pointed end (or conversely, the fossil pivots about the blunt heavier end which is thus up-stream or perhaps the reverse if that is the hydrodynamically stable position), glacial striations (the direction of advance can be ascertained from closer examination of the scratch), *flute marks* (groves formed on the sea floor in the direction of flow), the normal to a bedding surface (pointing up, say, towards the direction of more recent deposition, the "facing" or "younging" direction), and many others.

The present orientations will, in general, not be the ancient orientations because earth movements cause translation, rotation and deformation. This is the subject matter of *structural geology* which thus is interested in a further set of oriented quantities. The recent book by Ramsey (1967) is a recent comprehensive and readable reference. In a simple type of folding, the strata will look like a stack of sheets of corrugated iron and one would be interested in the orientation of the generator of this surface since it would be normal to the compressive movement. The tops and bottoms of the folds will have maximum curvature—lines joining points of maximum curvature on the same fold and same bed are called *minor fold hinge lines* or *fold axes*. There are several such lines in any one fold, for each bed or stratum. The surface containing them is called the *hinge surface* or *axial plane*. These lines and surfaces are approximately straight and planar respectively. Moreover, the hinge planes for nearby folds are approximately parallel. It is also found that the *cleavage planes* are approximately parallel to the hinge planes. Thus the hinge lines are approximately the lines of intersection of the cleavage planes and the *bedding surface*. *Planes of foliation* are essentially the same as cleavage planes—there is only a textural difference. Of course, a given formation may have been subjected to several generations of folding so its elucidation even if there are many *exposures* is not simple. There are clearly a number of lines and planes whose orientations must be measured.

The structural geologists have naturally taken over several concepts from the theory of elasticity, e.g. the *strain ellipsoid* which is defined at each point in a strained or deformed substance. A brief account of the theory is given in Section 6. Mathematically then, at each point, there is a 3×3 symmetric matrix or 2nd-order tensor. As one moves from point to point, the ellipsoid (in general) rotates and is distorted. This may sometimes be seen directly. If a spherical object were imbedded in the rock before straining and had the same mechanical properties as the rock, it would at all times have the shape of the strain ellipsoid. For example, on cleavage planes one may see the long and intermediate axes of an oolite.

Many other examples may be given of tensors in geology and crystallography. Fluid permeability, electric and magnetic susceptibility, stress, strain rates, thermal expansion and the optical indicatrix are all 2nd-order tensors in anisotropic substances. The book by Nye (1967) is an ideal mathematical reference.

2. ORIENTATION DEFINITIONS, MEASUREMENTS AND DATA PLOTTING

In Section 1, examples were given of various directional quantities:

(i) lines with a sense, i.e., unit vectors (e.g., direction of remanent magnetism, glacial striations, normal to a bedding plane in facing direction);

(ii) lines without a sense, i.e., axes (e.g., hinge line or fold

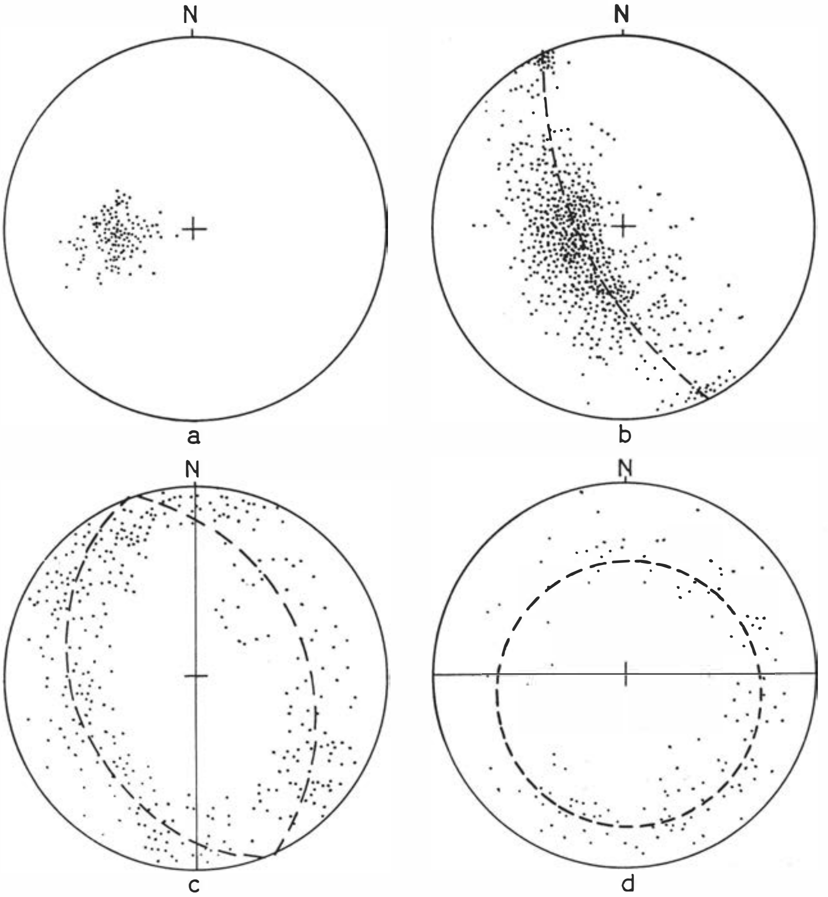

Fig. 2.1. Patterns of preferred orientation (median girdles show by broken lines). (*a*) Maximum (symmetric): 150 lineations from Loch Leven, Scottish Highlands. (*b*) Girdle: 1000 poles of foliation from Turoka, Kenya. (*c*) "Crossed girdle": 390 [0001] of quartz from quartzite, Barstow, California. (*d*) Small circle or "cleft" girdle: 140 [0001] of quartz from Orocopia schist, California. (After J. M. Christie; redrawn from Turner & Weiss (1963).)

axis, normal to a cleavage plane, normal to a foliation plane, direction of the *c*-axis in a calcite crystal);

(iii) tensors of order 2 or more (e.g., strain ellipsoid, magnetic susceptibility, elastic compliances).

In class (i), *unit vectors*, the quantity may be identified with a point on the surface of a sphere of unit radius. Sample data are naturally plotted this way for visual inspection—see Fig. 2.1 (*a*). A directed line, such as a glacial striation, which is measured in the field may be specified in a number of ways and the measurements are usually made with a Brunton Compass, which in addition to a magnetic needle has a level bubble and, essentially, a protractor. The angle between two vertical planes, through the sampling point, one containing the vector and one containing the magnetic north pole is the trend angle. It is given in degrees east of or clockwise from the north. It is sometimes called the *azimuth* or *declination*. The "*dip*" angle is the angle between the vector and the horizontal plane, taken as positive if the vector is below the horizontal plane. The "dip" is sometimes called the *inclination* or *plunge*. Thus a vector with trend t and dip d would have direction cosines on axes vertically downwards, northwards, eastwards of $\sin t$, $\cos t \cos s$, $\cos t \sin s$, respectively. Further specifications are given below.

The orientation of a plane is invariably defined by its strike and dip. The bubble allows the determination of a horizontal line in the plane. Choose the smaller of the two possible angles this line makes with north as the *strike* of the plane. Thus strikes run from $-90°$ (W) to $+90°$ (E). The *dip* of the plane is the dihedral angle with the horizontal plane; to avoid ambiguity dips must be described as "to the East" or "to the West". If the plane has a facing, a directed normal or pole may be defined. If a plane has strike s, and the up-face dips down to the west by an amount d, the direction cosines of the normal of the up-face relative to downwards, northwards, eastwards axes are

$(-\cos s \sin d, -\cos d, \sin s \sin d)$.

Directed lines are often in planes (e.g., striations). If the horizontal line in the plane and its strike is found, the angle between the directed line and the strike line (given its northern

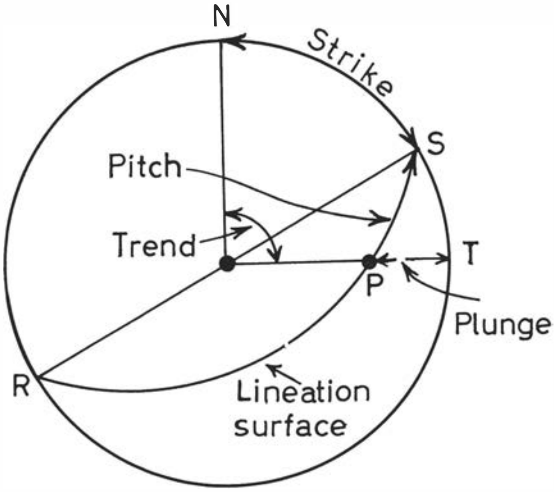

Fig. 2.2. The curve RPS is the intersection of a plane (the lineation surface) with the lower hemisphere. The orientation of a line in this plane is shown by OP. A vertical plane through the line cuts the hemisphere along the line OPT; this plane is, in practice, judged by eye. In terms of this Figure, the angles of pitch, plunge, strike, and trend are defined.

sense) is the *pitch* of the directed line. Pitch may be determined more accurately than trend because it is not necessary to estimate the imaginary vertical plane through the lineation. *Strike, dip and pitch* specify the direction uniquely. *Trend and plunge* or *declination and inclination* or *azimuth and "dip"* (pairing the words as they are used in different subjects) do so too. Note that dip is used in two senses so that we have used "dip" and dip; we will henceforth drop the azimuth and "dip" terminology. Declination and inclination is a usage restricted to magnetism so we will not use that either.

There are six combinations {(Strike, dip, trend), (Strike, dip, pitch), (Strike, dip, plunge), (Strike, trend, plunge), (Dip, trend, plunge), (Dip, trend, pitch)} which could be used to specify the orientation of a directed line. The combination which gives the greatest accuracy depends on the practical situation. The measurement errors in some of these quantities are correlated and dependent on the values of other quantities, e.g. when a plane is almost horizontal (low dip) its strike is hard to determine. While these problems are well known and intuitive "solutions" are part of the practical training of geologists, there do not appear to be any formal studies in print.

The various measurements mentioned above also suffice to orient undirected lines and planes which have no preferred "facing direction". Both these objects can be identified with the points on a hemisphere Fig. 2.1*b–d*.

We now turn to the practical question of plotting points on hemispheres and spheres and showing the results on a two-dimensional page. Clearly some projection must be used. The equal-area projection of a hemisphere is often

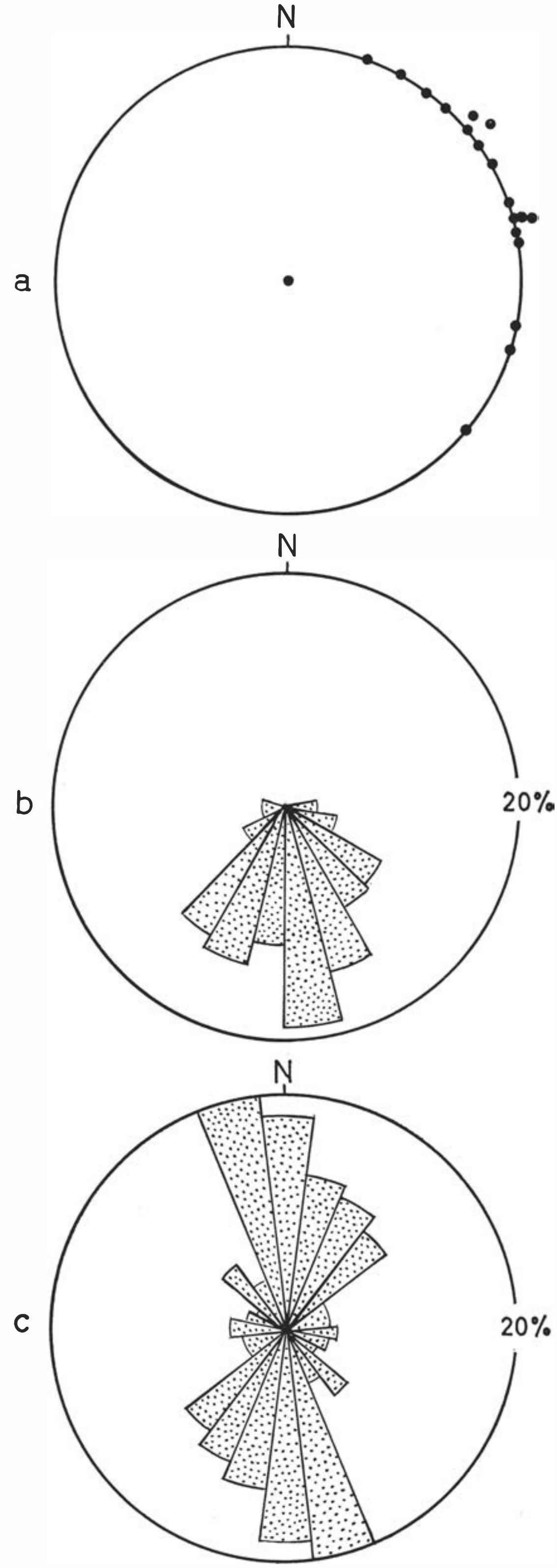

Fig. 2.3. (*a*) Raw data plotted on circle; (*b*) rose diagram of some palaeocurrent directions; (*c*) rose diagram of some axes of slump folds. (Redrawn from Potter & Pettijohn (1963).)

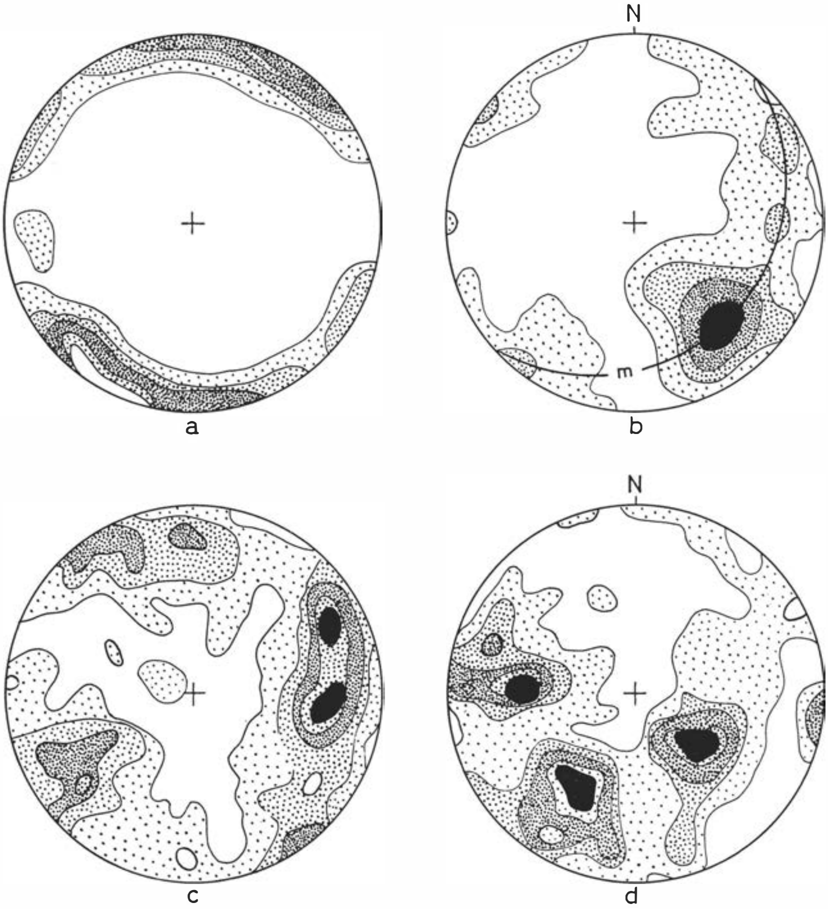

Fig. 2.4. Examples of monoclinic and triclinic subfabrics. (*a*) Monoclinic subfabric; 255 [0001] axes of quartz in mylonitized quartzite from Barstow, California. (After L. E. Weiss.) Contours, 9%, 7%, 5%, 3%, 1%, per 1% area. (*b*) Monoclinic subfabric; 2000 poles to foliation in schists and quartzites from Loch Leven, Scottish Highlands. (After L. E. Weiss.) Contours, 4%, 3%, 2%, 1% per 1% area. (*c*) Triclinic subfabric; 416 [0001] axes of quartz in deformed quartzite pebble from Panamint Range, California. (After L. E. Weiss.) Contours, 5%, 4%, 3%, 2%, 1%, per 1% area. (*d*) Triclinic diagram (heterogeneous body); 193 poles to foliation in gneiss and schist near Lake O'Keefe, East Central Quebec, Canada. (After G. Gastil & L. E. Weiss.) Contours, 5%, 4%, 3%, 2%, 1%, per 1% area. (Taken directly from Turner & Weiss (1963).)

used (i.e., the Lambert projection); it is easy to compare the density of plotted points in different regions. In the earth sciences this is known as the *Schmidt net* after W. Schmidt who first used it in structural geology in 1925. A very complete description of its practical usage is given in Turner & Weiss's (1963) text. Another suitable description is given in Ramsey (1967). The Wulff or stereographic net, which is always used in crystallography, has some advantages too since it is an equal angle, rather than an equal area, map. To understand such figures it must be realized that one is looking down into a hemisphere and that down here usually means also vertically downwards. North is at the top of the figure and East 90° clockwise from N at the right handside of the figure. There are 90° between the center and circumference of the figure. To plot unit vectors is no problem provided they all go into the bottom hemisphere. The angle between the lines from the center of the figure to N and to the point, is the trend angle. The distance of the point in from the circumference is the plunge angle. The points in Fig. 1 have been plotted this way. If they go into the upper hemisphere, the sign of the plunge is reversed and the points are marked with a different symbol (say, an open circle when the other points are filled circles). If the data gives a cluster of points above and below the perimeter, it us usually convenient to rotate the hemisphere so that they are nearer to the center. The intersection of a plane through the origin, which is vertically above the bottom of the hemisphere, and the hemisphere will be an arc. The line through the ends of the arc (where the arc cuts the circumference) and the center of the figure is the strike line of the plane. The greatest distance from the arc to the boundary is the dip angle. If a lineation (directed line) lies in this plane and points in the dipping direction, its corresponding point must be somewhere on the arc corresponding to the plane.

In fact the angular distance along the arc from the boundary to the point will be the pitch. Thus the relationship of all orientations measures is best shown as in Fig. 2.2.

It has been assumed above that all orientations were three-dimensional. Many are of course in two dimensions and here there are no complications—trend = strike may be plotted on the circumference of a circle. Then only directed and undirected lines are involved; for the latter it is usual to plot the *two* points where the line cuts the circle. A semi-circle could be used but the resulting picture is not as easy to interpret for obvious reasons.

Examples are shown in Fig. 2.3 of two-dimensional data. These figures immediately suggest that some kind of smoothing or histogramming will be necessary to estimate the underlying distribution in the population. On the circle most workers merely group according to the angle and draw a "rose-diagram", i.e. a histogram in which the usual boxes are sectors with angle equal to the angular class interval and radial length proportional to the class frequency. It is clear that most of the smoothing methods familiar to statisticians can be used without change.

Fig. 2.4 gives some examples of three-dimensional data. The book by Turner & Weiss (*ibid.*) gives many more and contains a list of manual methods for smoothing the data —referred to there as "contouring" since efforts are made to draw contours of equal density. The Mellis method described there could easily be programmed as it is a special case of the well-studied density estimator

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_n(1 - x_i' x)$$

where x is a point and $x_1, \ldots, x_n$ is the sample on the sphere and $\delta_n(1 - x_i'x) = \text{const.}$ for $x_i'x \geqslant c$, say and zero for $x_i'x < c$. Manually circles of an area depending on the sample size are drawn around each sample point and the areas traversed by 0, 1, 2, ... circular arcs are noted. The dependence on sample size and the possible use of more sophisticated functions $\delta_n(.)$ should be investigated statistically—this has never been done despite the fact that thousands of such diagrams must be drawn every year. The geological literature contains some heated discussions of the relative merits of different methods of contouring and the closely associated problems of testing for randomness or some other pattern. These discussions continue because of the lack of appreciation of basic statistical logic and theory.

The earlier sections in this chapter may have implied that only field measured data is plotted this way. In many cases however the specimens are oriented *in situ* but measured in the laboratory. In palaeomagnetism such is the case for an elaborate magnetometer is required. To measure the orientations of certain optical axes of the many small crystals at the surface of a specimen, ("petrofabrics") the surface must be prepared so it is smooth and flat and then placed in a laboratory machine.

Turning to the tensor quentities of orders greater than one, the usual situation would be the recording of the orientation of the specimen in the field and the laboratory measurement. For many second order tensors, the laboratory procedure is like that used to determine the magnetic susceptibility μ which has been statistically studied quite fully by Hext (1963). μ is a 3×3 symmetric matrix. If B (3×1) is the magnetic induction and H (3×1) is the magnetic field strength

$$B = \mu H.$$

If $B = \|B\| b, H = \|H\| h,$

then since

$$b'B = b'\mu H = b'\mu h \|H\|$$

$$\frac{\|B\|}{\|H\|} = b'\mu h.$$

Thus Hext assumes measurements of $\|B\|/\|H\|$ are made for various choices of b and h, the errors being independently, normal with zero mean and constant variance. The elements of μ are then found by least squares. The bulk of the paper is concerned with the interesting design problem which arises because the orientations of b and h can be chosen by the experimenter. The theory and results are similar to the response surface theory of Box & Hunter (1957). It will be shown in Section 6 that Hext's methods can often be used to estimate the strain tensor.

3. STATISTICAL METHODS

3.1. *Introduction*

The most difficult statistical problems in the earth sciences are concerned with collecting the data—getting random samples from well-defined populations. This is not surprising since the data of the earth sciences comes largely from field observation rather than from planned laboratory experiments. Except for several isolated topics (within and between site sampling and serial correlation), this basic area cannot be encompassed in this paper—it is not, of course, limited to orientation problems.

Some methods of plotting data for visual inspection have been mentioned in Section 2. For much work, very little else is used because of the lack of theories to predict the distribution of orientations. All plots must summarize the data collected to stimulate induction.

It is convenient in this introduction not to separate problems by dimensionality. Although the descriptions will refer to three (or more) dimensions, only trivial changes are required for them to make sense in two dimensions.

The commonest question is: is there a preferred orientation? That is: is it likely that the sample is a random sample from a uniform distribution? Many tests have been proposed and the controversy in the geological literature over their relative merits is not illuminated by reference to alternatives to uniformity (or randomness). A significant result is often used to justify a particular contouring. The references Stauffer (1966), Flinn (1958) will lead the reader to the geological literature on this point. Some more formal discussion of testing for uniformity is given below.

It is clear that many of the usual testing situations will arise. E.g., do two or more samples come from the same distribution? This suggests non-parametric methods. There will however be difficulties even on the circle with ordering the observations or forming a cumulative distribution function. To get optimum (in some sense) procedures one is lead to ask for parametric distributions as alternatives—see Section 3.5.

Some of the figures show data points in a single roughly circular cluster. Such samples, and presumably the populations they come from, can be well described by a single mean vector and some scalar measure of the scatter or dispersion about it. If the N observations are denoted by the unit vectors $r_1, \ldots, r_N$, it is very natural intuitively to use the direction of $R=\sum_1^N r_i$ (i.e. $R/\|R\|$ where $\|R\|$ is length of R) as the sample estimate of the mean direction. If the dispersion of the sample about $R/\|R\|$ is small, $\|R\|$ will be nearly as large as N. Thus $N-\|R\|$ makes a sensible measure of dispersion of the sample about its mean vector. In Section 3.2 it will be shown that if the probability density at r on the sphere is proportional $\exp(\mathcal{K}r'\mu)$, then samples would tend to be circularly dispersed about μ with a dispersion which varies inversely as $\mathcal{K}$. Furthermore the estimator of μ is $R/\|R\|$ and the estimator of $\mathcal{K}$ varies inversely with $N-\|R\|$. It is easy to make up other distributions which would lead circular dispersion about a single mean vector but none seem to lead to an easy development of statistical theory. Section 3.2 gives all the methods for $\exp(\mathcal{K}\cos\theta)$ in three dimensions; this case generalizes to more than three dimensions—Section 3.3 does the same for the circular case.

As is clear from the preceding sections, much data is axial and confusion may result from treating it as directional. Just as the center of mass analogue suggests methods for directional data, the analysis of axial data is motivated by the mechanical analogue of moments of inertia. To construct a suitable probability density, the fact that the sign of $\cos\theta = r'\mu$ should no longer matter leads us to consider densities proportional to $\exp(\mathcal{K}\cos^2\theta)$. When $\mathcal{K}>0$, the distribution is bipolar. When $\mathcal{K}<0$, it is a girdle distribution. Moreover the estimation procedures reduce to moment of inertia calculations, as hoped. Only one principal axis however has meaning. A more general distribution in which all three axes have singificance has density proportional to $\exp(\mathcal{K}_1(r'\mu_1)^2+\mathcal{K}_2(r'\mu_2)^2+\mathcal{K}_3(r'\mu_3)^2)$ where μ_1, μ_2, μ_3 are orthonormal. On the circle, these problems collapse and by a simple trick the methods of Section 3.3 become applicable. The whole subject is treated in Section 3.4.

Section 3.5 surveys non-parametric methods, many of them very recent. Section 3.6 summarizes Hext's work on second order tensors. Section 3.7 collects some miscellaneous papers.

The section as a whole attempts to cover all the well established methods. In the later sections, further methods and unsolved problems are suggested for future study.

3.2. *Unit vectors in three dimensions*

In this and the following subsections, some seventy pages of text has been eliminated by giving only the references in the order in which they appear and some sub-headings and notes. The full account will be available soon in another publication.

Purely random vectors: Daniel Bernoulli (1734), K. Pearson (1906), Lord Rayleigh (1919), Watson (1956*b*), Stephens (1964).

Large values of $\|R\|$ may be approximately tested by referring $\|R\|$ to $(N/3\,\chi_3^2(\alpha))^{\frac{1}{2}}$.

An appropriate test for Bernoulli's problem is that just given. The orbits of the nine planets are specified by i, the inclination of the orbital plane to the ecliptic, and Ω. To define Ω we take the positive side of the plane of the ecliptic as the direction of the thumb of the right-hand when the fingers point in the direction the earth moves in its orbit. Let a line be drawn through the sun to the point where the planet's orbit rises to the positive side of the ecliptic—the ascending node. Then Ω is the angle between these lines. The normals to the planetary orbit planes are naturally given sense by the right-hand screw rule. The data are, to the nearest minute of arc,

	i	Ω
Mercury	7°0′	47°08′
Venus	3°23′	75°47′
Earth	0°	0°
Mars	1°51′	48°47′

	i	Ω
Jupiter	1°19′	99°26′
Saturn	2°30′	112°47′
Uranus	0°46′	73°29′
Neptune	1°47′	130°41′
Pluto	17°10′	109°0′

The directions cosines of a typical normal are

$$r' = (\sin \Omega \sin i, \; - \cos \Omega \sin i, \cos i).$$

Thus $\Sigma r = R$ may be found and hence $\|R\|^2 = 50.85$. Since $N = 9$, $\|R\|^2$ is to be compared with $\frac{9}{3}\chi_3^2(\alpha)$. But χ_3^2 (0.001) = 16.268. Thus $\|R\|^2$ is just significant at the 0.001 level. The exact tables mentioned do not give the 0.001 level of $\|R\|^2$.

Although Arnold (1941) considered various distributions on the sphere including $\exp(\mathcal{K} \cos \theta)$, the statistical theory did not really begin until Fisher's (1953) paper which was motivated by the statistical problems of palaeomagnetism and a desire to discuss fiducial inference in a new example. As a consequence it is convenient to refer to

$$\frac{\mathcal{K}}{4\pi \sinh \mathcal{K}} \exp(\mathcal{K} r'\mu) \tag{3.2.3}$$

Fisher's distribution and the corresponding density on the circle as von Mises distribution. (The tendency to refer to them as the spherical and circular normal distributions is to be deplored.)

Watson (1956*a*), Watson & Williams (1956*c*), Watson (1960, 1965, 1966, 1967), Watson & Irving (1957). In these references a unified analysis of dispersion method is given for most problems arising from Fisher's distribution.

Beran & Watson (1967), Doell & Cox (1965), Ursell & Roberts. Test for serial correlation, diffusion approximation, robustness results.

3.3. *Unit vectors in two dimensions*

Kluyver (1906), K. Pearson (1906), Courant & Hilbert (1953), Greenwood & Durand (1955, 1958), von Mises (1918), Gumbel, Greenwood & Durand (1953), Batschelet (1965). The basic distribution theory of $\|R\|$ (approximately $(N/2X_2^2(\alpha))^{\frac{1}{2}}$ is given and the von Mises distribution introduced. Then follows the sampling theory of this distribution, largely from Watson & Williams (1967). Stephens (1962, 1963, 1965). Diffusion approximation and robustness results.

3.4. *Axial distributions*

Bingham (1964), Shelby (1964), Watson (1965, 1966), Krumbein (1939). As this literature is not at all clear, an attempt was made to give a commonsense account. The distributions used are exp $(\pm \mathcal{K} \cos^2 \theta)$ and Bingham's elaboration. On the circle, the theory of section 3.3 may be used if all observed angles are doubled since then 2θ and $2(\theta + \pi)$ coincide.

3.5. *Non-parametric procedures*

Kuiper (1960), Darling (1957), Stephens (1965), Stephens & Magg (1968), Watson (1961, 1962), Burr (1964). This discussion gives the circular analogues of the Kolmogorov and Cramér–von Mises tests. The theory of optimal tests grew suddenly in Ajne (1966), Schach (1967), Beran (1968 *a*, *b*), Watson (1967), Wheeler & Watson (1964). This theory is very attractive mathematically. One curious by-product is the alternative for which Watson's U^2 is optimal. Most work has been on the circle but Beran's work is much more general.

3.6. *The estimation of second order tensors*

Nye (1967), Hext (1963). If j, h are known unit vectors, ψ a symmetric matrix, it is assumed that the observation y is

$$y = j'\psi h + e,$$

where the errors are N.I.D. $(0, \sigma^2)$. The sampling errors in the least squares estimate of ψ are propagated into its spectral form. Tests and confidence regions are given.

3.7. *Miscellaneous*

Mackenzie (1967) proposed and solved elegantly the following problem. Suppose a number of known directions x_i are subject to an unknown rotation R and then measured as y_i. Assuming y_i is Fisher-distributed about Rx_i, R can be estimated, tests made, etc. Mackenzie & Thomson (1957) raised a quite different problem about the random disorientation of cubes.

4. PALAEOMAGNETISM

Some of the minerals present in rocks (usually iron oxides and sulfides) become magnetized in the presence of a magnetic field. The magnetism of such a rock depends on its mode of formation and subsequent history. Such a fossil or natural remanent magnetism (N.R.M.) may have been acquired when the rock cooled below its Curie point. It may also occur in sediments by the alignment of detrital particles containing iron. In either case, it records the local field at that time although subsequent geologic events may alter it.

The current magnetic field is complex and varying. Palaeomagnetic studies of lavas have shown however that if the earth's field is averaged over several thousand years it is well approximated by the field of a dipole at the center of the earth pointing along the spin axis of the earth. This means that the relationship of the point where the rock is examined and the spin axis, at time of formation may be found. However it is necessary to make reliability checks of many kinds, before such assertions can be made.

Some of these checks are of statistical interest. Graham (1949) suggested two. Suppose a bed contains pebbles derived from a rock formation being studied. If the magnetization of the parent formation and the pebbles has not been altered since it was put in, the magnetic orientation of the pebbles should be completely random. For the mechanical forces outweigh the magnetic forces. This will then be checked in practice by the test of randomness given in Section 3.2.

Graham also pointed out that if the formation of interest has been folded, a check on the permanence of the magnetization can be made by correcting the observed N.R.M.'s of samples for the rotations due to folding. The corrected sample of directions should then be a homogeneous cluster compared to the original data which would tend to be spread around a circle. It was by this method that the permanence of N.R.M. over geological periods was first shown. No significance test has been used or proposed. It would have to be based on a presumed initial distribution (e.g., Fisher's) for the sample.

This latter argument has been used (e.g., by Watson & Irving (1957)) to provide a chi-square test. If the rock contains grains of two sorts—one stably magnetized and one unstably magnetized (and thus likely to follow the variation of the earth's field)—non-circularly scattered samples may result from variations in composition from one sample to another. It has become quite standard to assume that good data may be treated as a random sample from Fisher's distribution. It should be remarked here, in passing, that this subject was fortunate, when it was in its infancy, to attract Fisher's attention.

Doell & Cox (1963) have criticized this universal assumption of Fisher's distribution. This led the author to make more extensive robustness calculations and to develop a test for non-randomness. Both have been described in Section 3.2. Doell & Cox sampled lavas in Hawaii on a (geologically) fine time scale. The slow movement of the field then made samples of similar dates more alike than those of very different dates. Harrison (1966), in studying deep-sea sediments, found that the directions of N.R.M. of adjacent samples in a long core were more alike than separated samples. Harrison proposed a vector version of von Neuman's (1941) statistic but did not find its distribution.

When studying a very large formation there are clearly sampling problems. It is natural to suppose that the mean N.R.M. varies from site to site across the formation and that different samples from the same site will vary. This being so, it is clear that it is better to take a few samples from many sites than the reverse. Violations of this precept have occurred and, further, the information contained in site to site variation was first lost by some workers who pooled all their data. This led Watson & Irving (1957) to suggest a within and between analysis—see Section 3.2. for the theory. The necessity of using weighted means, and optimizing the sampling, when the scatters of different sites are very different is also pointed out in this article.

Given a sample of N.R.M. directions at a point and that all the scientific assumptions are valid, it is easy to relate the observed mean palaeomagnetic direction to the predicted ancient pole position. The axis of the dipole must lie in the plane through the center of the earth and the palaeolatitude λ and the inclination I of the mean direction are connected by

$$\tan I = 2 \tan \lambda.$$

A confidence region for this pole position is found by predicting a pole for every direction in the cone of confidence of the estimated palaeomagnetic direction. As mentioned in Section 3.2, this procedure is due to Fisher (1953). Numerical examples and fuller discussions are given in the original papers, and by Irving in his book.

Palaeomagnetic studies soon showed two exciting features. Firstly the pole predicted from rocks of different ages in the same continent was found to wander. The wandering curves were not the same for different continents. This led to the rehabilitation of the Continental Drift hypothesis of Wegener (1924) and further to reconstructions of where the various continental masses were at different times. Much more evidence has now been obtained. Evidence from new radiometric dating methods was recently described by Hurley (1968).

The other anomaly noticed was that the earth's field has reversed a number of times. It was first thought that this effect might not be real but be a consequence of solid state phenomena. The reversals are now accepted and are proving useful in providing a time scale for sea-floor spreading, for example, see Cox, Dalrymple & Doell (1967). The discovery of the mid-ocean ridges led to the notion that material is being convected up to the ocean bottom, literally pushing the continents apart. As it cools it will take the local field direction at that time. Samples from the ocean bottom showed bands of reversed polarity. To put on a time scale for this possible source of continental drift, it was necessary to date recent (surface) lava flows using a potassium-argon method. A very interesting statistical problem arises if one tries to make this scale more precise and objective. Since the bands are classified only as normally or reversely magnetized, there is no error in the directional measurement. The errors lie only in the isotopic dating method. In the vicinity of the transition (say, from normal to reverse) time, there will be conflicts in the data which might have the form $N\,N\,N\,R\,N\,R\,R\,N\,R\,R\,R$. Since each new sample brings a new unknown—its true age—standard statistical methods fail. Cox & Watson (1968) give several procedures.

A related field has made less progress. Runcorn suggested that since the wind systems depend on the rotational poles, a study of aeolian deposits should be of interest in determining ancient wind directions. Analysis of barchans requires similar statistical methods. A general description is given by Opdyke & Runcorn (1959).

In conclusion it should be made clear that there are many other statistical problems in this subject and so more or less related to directions. The study of the accuracy parameter $\mathcal{K}$ of Fisher's distribution for different sediments is one example the reader may well think of. The suggestion of the author that Fisher's distribution could arise as a Boltzman distribution in this case did not stand up. At a more speculative level stochastic models have been considered for some of the phenomena. The subject has fortified beliefs that the earth's field is due to convection in the mantle. The westerly drift of the field is attributed to the relative motion of the crust to the interior of the earth. Fluctuations like reversals must then be due to instabilities in the convection cells. This raises the general question of a dynamical system with two stable states subject to random impulses in some way—can one say anything about the distribution of transition times from orbits about one state to orbits about another? Another speculative question of interest concerns the earth's radius in ancient times. To accommodate some continental reconstructions, the ancient radius needs to be larger. It is graphically clear that the method used to calculate old pole positions assumes a constant earth-radius. Ward (1963) tried to estimate the palaeoradius by choosing it to minimize the dispersion of ancient poles. There are a variety of other methods and estimates and so the problem remains contentious. A recent review of many geophysical problems and their extensions to the planets is Runcorn (1967).

5. SEDIMENTARY GEOLOGY

The deposition of sediments leads inevitably to anisotropy and orientation problems, as mentioned in Section 1, abound in this part of Geology. Figs. 5.1, 5.2, taken from Potter & Pettijohn (1963) show this very simply. In each case an axis is involved (normal to a disc-like stone, axis of an cigar-shaped stone) and the points where axes hit the lower hemisphere are shown. This is an example of the plotting mentioned in Section 2. Two extreme pebble shapes are used. In practice, problems of definition arise. The normal to the maximum projection plane is sometimes used when stones are not simple discs. Long axes of pebbles are defined in various ways. The same definitions are used for fossils which are oriented by the same processes. On a smaller scale (sand-sized particles) the principles are the same but measurement difficulties have led to changes in tactics, e.g. an optic axis of a quartz grain may be used instead of its longest axis, although they need not coincide, because it is easier to measure accurately. It is possible that the use of more inaccurate measurements is a better strategy. These difficulties have also encouraged the measurement of various tensor quantities like fluid permeability and magnetic susceptibility. In the case of the former, the directions of greatest,

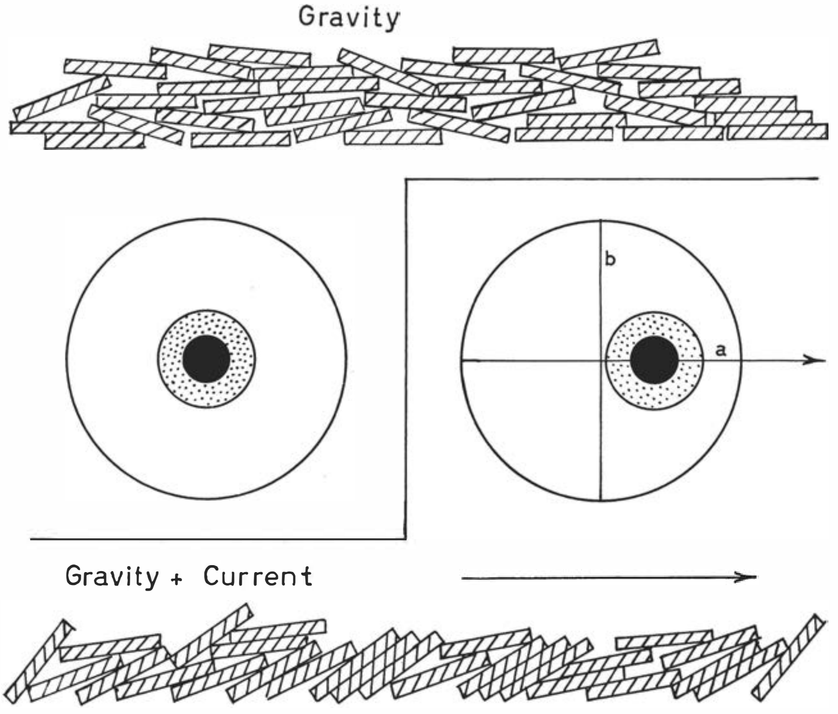

Fig. 5.1. Orientation of disks by gravity (orthorhombic symmetry) and by gravity plus current action (monoclinic symmetry). (Redrawn from Potter & Pettijohn (1963).)

intermediate, least flow are the principal axes or eigenvectors of the tensor corresponding its greatest, intermediate and least eigenvalues. Efforts to show that average orientation of the long axes of grains is related to the direction of greatest permeability

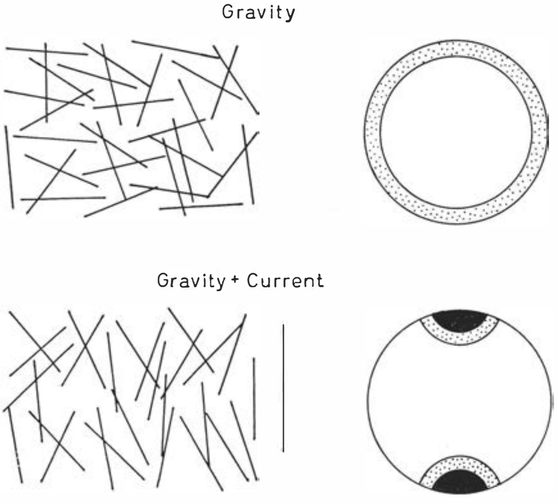

Fig. 5.2. Orientation of rods by gravity and by gravity plus current action. (Redrawn from Potter & Pettijohn (1963).)

and the bedding plane have, apparently, been unsuccessful. With elongated ferrimagnetic grains, the average long axis does seem to be along the principal axis for greatest magnetic susceptibility. The study of remanent magnetism, mentioned in Section 4, in sediments could throw some light on the deposition process. Fluid velocity would interfere with the alignment of the particles with the local magnetic field, either increasing the dispersion, or reducing the intensity of the magnetism. Not much is known, however.

Most information on palaeocurrents come from the study of ripplemarks and cross-bedding. A very detailed study of the effects of water running over a river or sea bottom and the effects of moving and stationary objects on the bottom surface may be seen, with many fascinating photographs, in Dzulynski & Walton (1965). The various measurable effects of ice-flow were mentioned in Section 1. Measurements of these features provide both directional and axial data.

It is clear that the methods mentioned in Section 3 will all find application. It is not to be expected that they will solve all the problems. There are, as one

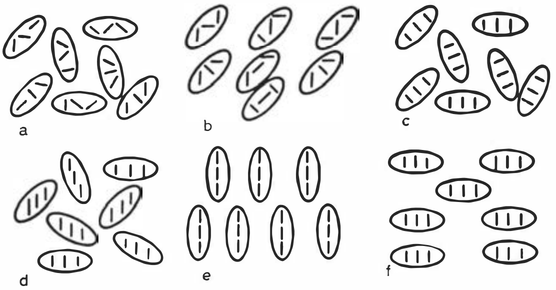

Fig. 5.3. Diagrammatic relations between internal directional structures and orientation of elongate sand bodies. (Redrawn from Potter & Pettijohn (1963).)

might guess, "mixed" problems. Fig. 2.3 (*b*), (*c*) shows a two-dimensional situation where a test is required that the mean axis of an axial distribution is parallel to the mean direction of a directional distribution. Of more interest perhaps is the isolation of new problems in the literature.

The discussion in Chapter 7 of Potter & Pettijohn, from which Fig. 5.3 is taken, suggests an apparently new problem. Elongated bodies may be orientated preferentially or randomly. Within these bodies are often directional features which, when sampled and measured, have orientations. There is a set of possibilities.

(i) bodies randomly oriented
  (*a*) features random w.r.t. bodies
  (*b*) features non-random w.r.t. bodies.

(ii) bodies non-randomly oriented
  (*a*) features random w.r.t. bodies
  (*b*) features non-random w.r.t. bodies.

Cases (i) *a*, (ii) *a*, mean that the features are randomly oriented with respect to fixed axes. In cases (i) *b*, (ii) *b*, the features could have a strong orientation w.r.t. fixed cases. Case (ii) *b* resembles the "variance-component" model behind Table 3.2.1 in Section 3.2 if only the measurements of the features are considered. Here, however, we would have orientation measurements on the bodies; these would be used to decide between (i) and (ii) using the Section 3 methods. To decide the (*a*) or (*b*) question, the feature orientations may be treated similarly. In the (*b*) situation, the features would then be given orientations relative to fixed axes and restudied. Thus it seems that no new problem arises.

The later Chapters of Potter & Pettijohn raise the statistical problem of mapping. The expression "trend surface" is used in geology for the result of smoothing observations to make a coherent map. As noted earlier this is a general problem of great importance but outside our scope since it is only marginally connected with orientations. They conclude with chapters on data collection, data displays and statistical analysis. They give references to the main papers read, and methods used, by geologists. Some of the methods are incorrect and should be replaced by the methods given in Chapter 3. A recent book by Griffiths (1967) gives much more attention to statistical questions, especially sampling problems.

6. STRUCTURAL GEOLOGY

6.1. *Introduction*

In Section 1, brief definitions were given of the main lines and surfaces of interest to the structural geologist in the field. The reduction of this type of data requires only the direct application of the methods of section 3, and so will not be pursued here (two examples are shown in Watson (1965)). Of course, the picture suggested in Section 1 of the folded layers resembling a stack of sheets corrugated iron is far too simplified. Ramsey (1967, Chapter 8) describes many kinds of folds. An integral part of the discussion there is, naturally, the origin of observed folds. It is intuitively clear that compression might be a common cause. The theory of buckling of elastic bodies goes back to Euler. The slow speed and circumstances in geology make non-elastic theory more relevant, e.g., it may be more pertinent to regard the rock as layered viscous material. This suggests other instability mechanisms for generating wave-like effects. One of the most characteristic of all wave phenomena is actually observed—interference between folds. The theories of Ramberg & Biot (see Ramsey for references) along these lines seem to cry out for the application of spectral analysis.

Regardless of the physics of the situation the movement and deformation, especially the latter, of the rocks is of interest. Deformation is usually referred to as *strain*. There is space here only to mention the simplest case—*homogeneous* strain—in which a point, initially at x, is at Bx after deformation. Here B is a constant matrix. There are cases where the matrix can be estimated directly, e.g., the positions of stakes in the central portion of a glacier could be found at several times. Because the deformation must grow from zero (i.e., B is initially equal to I) and physics requires the transformation always to be 1–1, we may assume $|B|>0$. Two points P and Q initially separated by a vector $Z_0 = PQ$ of length s_0 are finally separated by s where

$$\frac{s^2 - s_0^2}{s_0^2} = \frac{z_0'(BB'-I)z_0}{z_0' z_0}; \tag{6.1.1}$$

of course the orientation as well as the length of PQ usually changes. The *strain matrix* or tensor e defined by

$$e = \tfrac{1}{2}(BB'-I), \tag{6.1.2}$$

is usually of more interest than the matrix B. If the rock contained a sphere of radius c which was mechanically indistinguishable from the body of the rock, then after deformation it would be an ellipsoid

$$y'(BB')^{-1}y = c^2. \tag{6.1.3}$$

Knowledge of this strain ellipsoid is equivalent to knowledge of e and easier to grasp intuitively. The serious reader should read the book by Ramsey for more details and motivation. However we can now pose, in Section 6.2, one of the major problems of this area—the estimation of the strain tensor.

6.2. *Strain determination at a point*

In geology, the strain at a point must be determined, not from before and after measurements on the rock mass, but from the deformation of objects embedded in the rock. Both the objects and the rock are opaque so that data from plane sections must be used. Useful information may be obtained from

(i) lines of known initial length,
(ii) angles of known initial size,
(iii) spherical, or nearly spherical bodies.

The results of deformation are vividly described with illustrations in Ramsey's book. As an example of (i), consider a crystal of tourmaline embedded in rock which is then stretched in the direction of this long narrow crystal. The crystal will break into a number of pieces if the strain is large enough. The initial length is, to a good approximation the sum of the lengths of the fragments. The final length is the length between the beginning of the first fragment and the end of the last fragment. An example of (ii) would be a brachiopod whose shape is roughly a half-circle. When deformed, the radius perpendicular to the base becomes inclined at a measurable angle from the base. It is estimated that 70% of all strain determinations are made by type (iii) studies. Oolites (concretory structures) are a common source of data. They are known to be roughly spherical initially and to have sizes of the order of 2 mm diameter. Sections in undeformed rock will then show circles of varying radii. In deformed rock, the sections show ellipses. There has been no statistical discussion of any of these problems in geology. The writer and Dr D. Elliot have a number of results which will be published elsewhere. There is space here only for a few comments.

The reduction of type (i) data can be reduced to an application of Hext's work, described in Section 6.3. Typical data of type (ii) comes from the deformation of brachiopods. It consists of N pairs of unit vectors which were, before deformation, orthogonal. Good methods are hard to find even in two dimensions. The common type (iii) data presents many problems, mostly soluble. The axial ratios of an ellipsoid will be underestimated by plane section measurements. How then should these be treated? Even when one can cut in the plane of least and greatest axes, and one assumes that the bodies are all initially spherical although of unknown radii, the data plot leads to "fitting a straight line when both variables are subject to error"—thus detailed knowledge is necessary. Depending on circumstances we have several solutions. It is, in fact, rare that the objects are intially spherical. Dr Elliott has a treatment of this problem awaiting publication. Further theory has been developed with the writer. Some interesting problems arise here of the general type—what is the effect of a linear transformation on a distribution, in space, of orientations?

It is obvious that, in these problems, the complete matrix e can only rarely be estimated—only some aspects of its spectral form. There are however some occasions when it can be. One then often has to combine estimates of e which have been made with respect to coordinate axes which are unknown rotations of some standard set of axes. In two dimensions, the use of the *Mohr-representation* makes this a problem of estimating the point of intersection of several lines, all in error.

7. PETROFABRICS

In Section 6, we considered some of the statistical problems which arise in structural geology when features such as folds are considered. In this section we will discuss problems related to small scale features, such as the orientation of optic axes of small crystals in the body of a rock.

A rock is usually an aggregate of grains of varying compositions, often crystals. The term *fabric* (or texture) is used to describe the distribution and orientation of these grains. To define what is meant by homogeneous, or statistically homogeneous, rocks, one must first select the quantities of interest and agree on a scale. Here we will take the scale to be much larger than the grain dimensions. Then there is, intuitively, sense in talking about homogeneous (strict or statistical) fabrics. Fabrics are of interest in structural geology because something of the origin and history of the rock is recorded in its fabric. The orientation aspects of the fabric of sedimentary rocks is generally much simpler than those of *tectonites*—sedimentary or igneous rocks which have been subjected to solid flow. The examples below therefore refer to tectonities. The classic reference is the book by Turner & Weiss (*ibid.*).

Petrofabric diagrams are made by measuring the orientation of some specified (usually optic) axis of every grain exposed on a thin section. The data is plotted by the methods of Section 2. Various "contouring" methods are used to display the data in a smoothed form. The significance of clusters and girdles of points may be tested against various null hypotheses by the methods of Section 3 and other methods. However, there is so little theory that there is rarely a specific alternative hypothesis. Geologists have therefore been rightly critical of testing—see e.g. Flinn (1958), Stauffer (1966). The null hypothesis is usually complete randomness.

In the above discussion, no account is taken of the position of the grains on the surface of the section. The exposed grains $G_1, \ldots, G_N$, say, as seen in Fig. 7.1, have almost arbitrary shapes and sizes. With each G_i is associated a direction or axis r_i, $i = 1, \ldots, N$. To discover whether grains with similar orientation are uniformly distributed over the section or tend to be clustered in some manner, *axial distribution analysis* (usually referred to as A.V.A., the initials of the German equivalent) was introduced by Ramsauer, a student of Sander.

To make an A.V.A., traverses of the section are made and the orientation of each grain determined. These are plotted on a petrofabric diagram and contoured. The resulting diagram is divided up into say, C areas so that the observed directions are put into C classes—the considerations used to form

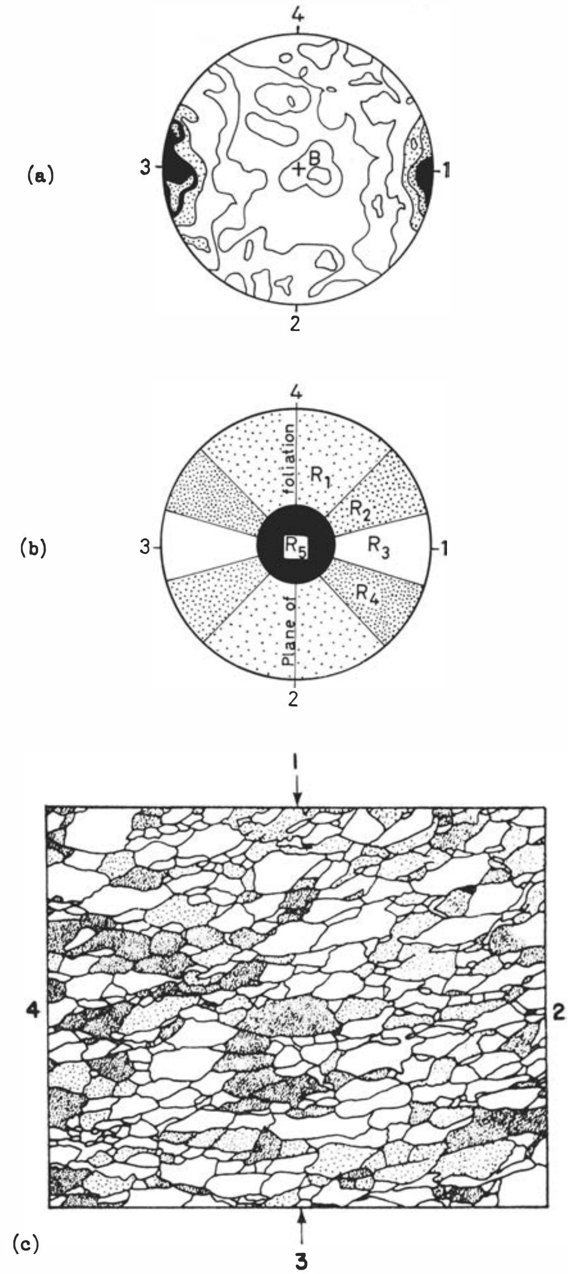

Fig. 7.1. When the orientation associated with each small domain in (*c*) is plotted and contoured, (*a*) is obtained. If orientations are grouped and shaded as in (*b*), the shaded picture (*c*) is obtained. (Redrawn from Turner & Weiss (1963).)

classes are those relevant to forming any histogram. A photograph of the section is then colored with C colors, a grain having color C_1 if its orientation is in the first class, etc. The resulting picture will obviously show up any strong tendency of neighboring grains to have similar orientation—see Fig. 7.1.

The problem raised is clearly a general one of *mapping orientations* and in no way restricted to petrofabric analysis. Equally the question of whether the orientations are spatially incoherent or correlated, is of general interest. Both were mentioned earlier in Sections 2 and 3. Further, once the directions are put into classes, the problem is quite general and not restricted to orientation work.

For the A.V.A. problem, Flinn (1965) suggested a test of the null hypothesis that the grain orientations are incoherent, i.e., that the colors observed in traverses across the A.V.A. diagram follow each other randomly. Let f_{ij} be the number of times color j comes immediately after color i, and let these frequencies be put in a $C \times C$ table whose rows and columns refer to the C colors in the same order. Because of the traversing necessary to scan the whole section, the row total $f_{i.}$ and the column total $f_{.j}$ may not agree exactly. The statistic used by Flinn is

$$X^2 = \sum_{i,j} \frac{\left(f_{ij} - \frac{f_i \cdot f_j}{N}\right)^2}{\frac{f_i \cdot f_j}{N}}, \quad N = \text{number of grains}.$$

It is familiar from the similar test for independence in Markov chains. At the suggestion of B. Selby, X^2 is referred to the chi-square table with $(C-1)^2$ degrees of freedom, as it is for Markov chains. Flinn, in discussing his test, implies that a petrofabric diagram however non-random it may appear is not geologically significant unless the associated A.V.A. is statistically significant. He also points out that his test will detect cases where, e.g., like follows unlike color too often, not only like following like colors. A non-significant result fortifies belief that the fabric, at this scale, and in this respect, is statistically homogeneous. At a somewhat larger scale, it would be quite possible to have a statistically homogeneous rock contain clusters of similarly oriented grains. (When in layers, these are called fein lagen.)

To avoid the grouping of directions into classes, it would be possible to use the test of Beran & Watson (section 3.3) if the traverses were put end on end—for then $\sum_1^N r_i' r_{i+1}$ can be calculated. Provided there were many more grains than traversers, this procedure seems quite reasonable and might be more sensitive because it does not group.

Both these methods are very sensitive to the traversing direction and separation. Flinn remarks: "The traverses are separated by a constant distance not less than an average grain diameter. If the grains have a strong shape orientation the traverses are best arranged to run normal to the average direction of the longest grain axes." It would thus be good to have a test not requiring traversing at all. With grouped directions, tests depending on the number of similarly colored grains which surround each grain come to mind. They are complicated by the varying number of grains surrounding each grain. Ideally one would prefer also not to group directions. Thus there is room here for statistical contributions.

8. ACKNOWLEDGEMENT

It would not have been possible for me to write this paper without the continued help of my colleague Dr David Elliott with the definitions of geological terms, the clarification of geological ideas, and with references. Section 2 is my version of one of his lectures in our joint seminar on "Statistical Problems in Geology". I am very grateful to him for the hours we spent trying to reduce the geological probiems mentioned above to forms with sufficient mathematical structure to be understood and resolved by applied mathematicians. This is one of the aims of the newly created Department of Earth and Planetary Sciences at the Johns Hopkins University, of which we are both members.

Sommaire. Les géosciences cherchent à décrire et à comprendre les processus du passé et du présent qui ont formé et forment encore les continents et les océans avec leurs montagnes et leurs vallées et qui sont l'origine des multiples séries de rochers de différentes structures et compositions. Avec le temps, ces processus transportent des matériaux d'un point à l'autre et, de cette façon, intéressent au point de vue de la direction. Il est donc naturel que, pour démêler ce qui se passe, on essaie de déterminer les directions. En cet écrit, nous voudrions donner aux statisticiens une vue générale des problèmes qui surgissent, dans l'espoir que d'autres seront amenés à s'y intéresser. Notre écrit contient plus de science que de statistique. Il couvre moins que son titre; ainsi, l'océanographie et la météorologie ont été exclues.

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